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## Research Article

# A Common Fixed Point Theorem in $D^*$ -Metric Spaces

Shaban Sedghi, Nabi Shobe, and Haiyun Zhou

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We give some new definitions of  $D^*$ -metric spaces and we prove a common fixed point theorem for a class of mappings under the condition of weakly commuting mappings in complete  $D^*$ -metric spaces. We get some improved versions of several fixed point theorems in complete  $D^*$ -metric spaces.

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## 1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [1] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [2], Erceg [3], Kaleva and Seikkala [4], and Kramosil and Michálek [5] have introduced the concepts of fuzzy metric spaces in different ways. George and Veeramani [6] and Kramosil and Michálek [5] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connection with both string and  $E$ -infinity theories which were given and studied by El Naschie [7–10]. Many authors [11–17] have studied the fixed point theory in fuzzy (probabilistic) metric spaces. On the other hand, there have been a number of generalizations of metric spaces. One of such generalizations is generalized metric space (or  $D$ -metric space) initiated by Dhage [18] in 1992. He proved the existence of unique fixed point of a self-map satisfying a contractive condition in complete and bounded  $D$ -metric spaces. Dealing with  $D$ -metric space, Ahmad et al. [19], Dhage [18, 20], Dhage et al. [21], Rhoades [22], Singh and Sharma [23], and others made a significant contribution in fixed point theory of  $D$ -metric space. Unfortunately, almost all theorems in  $D$ -metric spaces are not valid (see [24–26]).

## 2 Fixed Point Theory and Applications

In this paper, we introduce  $D^*$ -metric which is a probable modification of the definition of  $D$ -metric introduced by Dhage [18, 20] and prove some basic properties in  $D^*$ -metric spaces.

In what follows  $(X, D^*)$  will denote a  $D^*$ -metric space,  $\mathbb{N}$  the set of all natural numbers, and  $\mathbb{R}^+$  the set of all positive real numbers.

*Definition 1.1.* Let  $X$  be a nonempty set. A generalized metric (or  $D^*$ -metric) on  $X$  is a function,  $D^* : X^3 \rightarrow [0, \infty)$ , that satisfies the following conditions for each  $x, y, z, a \in X$ :

- (1)  $D^*(x, y, z) \geq 0$ ,
- (2)  $D^*(x, y, z) = 0$  if and only if  $x = y = z$ ,
- (3)  $D^*(x, y, z) = D^*(p\{x, y, z\})$ , (symmetry) where  $p$  is a permutation function,
- (4)  $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$ .

The pair  $(X, D^*)$  is called a generalized metric (or  $D^*$ -metric) space.

Immediate examples of such a function are

- (a)  $D^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$ ,
- (b)  $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ .

Here,  $d$  is the ordinary metric on  $X$ .

- (c) If  $X = \mathbb{R}^n$  then we define

$$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p} \quad (1.1)$$

for every  $p \in \mathbb{R}^+$ .

- (d) If  $X = \mathbb{R}$ , then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max \{x, y, z\} & \text{otherwise.} \end{cases} \quad (1.2)$$

*Remark 1.2.* In a  $D^*$ -metric space, we prove that  $D^*(x, x, y) = D^*(x, y, y)$ . For

- (i)  $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$  and similarly
- (ii)  $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$ .

Hence by (i), (ii) we get  $D^*(x, x, y) = D^*(x, y, y)$ .

Let  $(X, D^*)$  be a  $D^*$ -metric space. For  $r > 0$ , define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}. \quad (1.3)$$

*Example 1.3.* Let  $X = \mathbb{R}$ . Denote  $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$  for all  $x, y, z \in \mathbb{R}$ . Thus

$$\begin{aligned} B_{D^*}(1, 2) &= \{y \in \mathbb{R} : D^*(1, y, y) < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| + |y - 1| < 2\} \\ &= \{y \in \mathbb{R} : |y - 1| < 1\} = (0, 2). \end{aligned} \quad (1.4)$$

**Definition 1.4.** Let  $(X, D^*)$  be a  $D^*$ -metric space and  $A \subset X$ .

- (1) If for every  $x \in A$ , there exists  $r > 0$  such that  $B_{D^*}(x, r) \subset A$ , then subset  $A$  is called open subset of  $X$ .
- (2) Subset  $A$  of  $X$  is said to be  $D^*$ -bounded if there exists  $r > 0$  such that  $D^*(x, y, y) < r$  for all  $x, y \in A$ .
- (3) A sequence  $\{x_n\}$  in  $X$  converges to  $x$  if and only if  $D^*(x_n, x_n, x) = D^*(x, x, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . That is, for each  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n \geq n_0 \implies D^*(x, x, x_n) < \epsilon (*). \quad (1.5)$$

This is equivalent; for each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that

$$\forall n, m \geq n_0 \implies D^*(x, x_n, x_m) < \epsilon (**). \quad (1.6)$$

Indeed, if  $(*)$  holds, then

$$D^*(x_n, x_m, x) = D^*(x_n, x, x_m) \leq D^*(x_n, x, x) + D^*(x, x_m, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad (1.7)$$

Conversely, set  $m = n$  in  $(**)$ , then we have  $D^*(x_n, x_n, x) < \epsilon$ .

- (4) A sequence  $\{x_n\}$  in  $X$  is called a Cauchy sequence if for each  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that  $D^*(x_n, x_n, x_m) < \epsilon$  for each  $n, m \geq n_0$ . The  $D^*$ -metric space  $(X, D^*)$  is said to be complete if every Cauchy sequence is convergent.

Let  $\tau$  be the set of all  $A \subset X$  with  $x \in A$  if and only if there exists  $r > 0$  such that  $B_{D^*}(x, r) \subset A$ . Then  $\tau$  is a topology on  $X$  (induced by the  $D^*$ -metric  $D^*$ ).

**LEMMA 1.5.** Let  $(X, D^*)$  be a  $D^*$ -metric space. If  $r > 0$ , then ball  $B_{D^*}(x, r)$  with center  $x \in X$  and radius  $r$  is open ball.

*Proof.* Let  $z \in B_{D^*}(x, r)$ , hence  $D^*(x, z, z) < r$ . Let  $D^*(x, z, z) = \delta$  and  $r' = r - \delta$ . Let  $y \in B_{D^*}(z, r')$ , by triangular inequality we have  $D^*(x, y, y) = D^*(y, y, x) \leq D^*(y, y, z) + D^*(z, x, x) < r' + \delta = r$ . Hence  $B_{D^*}(z, r') \subseteq B_{D^*}(x, r)$ . Hence the ball  $B_{D^*}(x, r)$  is an open ball.  $\square$

**Definition 1.6.** Let  $(X, D^*)$  be a  $D^*$ -metric space.  $D^*$  is said to be a continuous function on  $X^3$  if

$$\lim_{n \rightarrow \infty} D^*(x_n, y_n, z_n) = D^*(x, y, z) \quad (1.8)$$

whenever a sequence  $\{(x_n, y_n, z_n)\}$  in  $X^3$  converges to a point  $(x, y, z) \in X^3$ , that is,

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y, \quad \lim_{n \rightarrow \infty} z_n = z. \quad (1.9)$$

**LEMMA 1.7.** Let  $(X, D^*)$  be a  $D^*$ -metric space. Then  $D^*$  is a continuous function on  $X^3$ .

*Proof.* Suppose the sequence  $\{(x_n, y_n, z_n)\}$  in  $X^3$  converges to a point  $(x, y, z) \in X^3$ , that is,

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y, \quad \lim_{n \rightarrow \infty} z_n = z. \quad (1.10)$$

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Then for each  $\epsilon > 0$  there exist  $n_1, n_2$ , and  $n_3 \in \mathbb{N}$  such that  $D^*(x, x, x_n) < \epsilon/3 \forall n \geq n_1$ ,  $D^*(y, y, y_n) < \epsilon/3$  for all  $n \geq n_2$ , and  $D^*(z, z, z_n) < \epsilon/3 \forall n \geq n_3$ .

If we set  $n_0 = \max\{n_1, n_2, n_3\}$ , then for all  $n \geq n_0$  by triangular inequality we have

$$\begin{aligned}
 D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) \\
 &\leq D^*(x_n, z, y) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\
 &\leq D^*(z, y, x) + D^*(x, x_n, x_n) + D^*(y, y_n, y_n) + D^*(z, z_n, z_n) \\
 &< D^*(x, y, z) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x, y, z) + \epsilon.
 \end{aligned} \tag{1.11}$$

Hence we have

$$\begin{aligned}
 D^*(x_n, y_n, z_n) - D^*(x, y, z) &< \epsilon, \\
 D^*(x, y, z) &\leq D^*(x, y, z_n) + D^*(z_n, z, z) \\
 &\leq D^*(x, z_n, y_n) + D^*(y_n, y, y) + D^*(z_n, z, z) \\
 &\leq D^*(z_n, y_n, x_n) + D^*(x_n, x, x) + D^*(y_n, y, y) + D^*(z_n, z, z) \\
 &< D^*(x_n, y_n, z_n) + \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = D^*(x_n, y_n, z_n) + \epsilon.
 \end{aligned} \tag{1.12}$$

That is,

$$D^*(x, y, z) - D^*(x_n, y_n, z_n) < \epsilon. \tag{1.13}$$

Therefore we have  $|D^*(x_n, y_n, z_n) - D^*(x, y, z)| < \epsilon$ , that is,

$$\lim_{n \rightarrow \infty} D^*(x_n, y_n, z_n) = D^*(x, y, z). \tag{1.14}$$

□

**LEMMA 1.8.** *Let  $(X, D^*)$  be a  $D^*$ -metric space. If sequence  $\{x_n\}$  in  $X$  converges to  $x$ , then  $x$  is unique.*

*Proof.* Let  $x_n \rightarrow y$  and  $y \neq x$ . Since  $\{x_n\}$  converges to  $x$  and  $y$ , for each  $\epsilon > 0$  there exist  $n_1, n_2 \in \mathbb{N}$  such that  $D^*(x, x, x_n) < \epsilon/2 \forall n \geq n_1$  and  $D^*(y, y, x_n) < \epsilon/2 \forall n \geq n_2$ .

If we set  $n_0 = \max\{n_1, n_2\}$ , then for every  $n \geq n_0$  by triangular inequality we have

$$D^*(x, x, y) \leq D^*(x, x, x_n) + D^*(x_n, y, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{1.15}$$

Hence  $D^*(x, x, y) = 0$  which is a contradiction. So,  $x = y$ . □

**LEMMA 1.9.** *Let  $(X, D^*)$  be a  $D^*$ -metric space. If sequence  $\{x_n\}$  in  $X$  is convergent to  $x$ , then sequence  $\{x_n\}$  is a Cauchy sequence.*

*Proof.* Since  $x_n \rightarrow x$ , for each  $\epsilon > 0$  there exists  $n_0 \in \mathbb{N}$  such that  $D^*(x_n, x_n, x) < \epsilon/2 \forall n \geq n_0$ . Then for every  $n, m \geq n_0$ , by triangular inequality, we have

$$\begin{aligned} D^*(x_n, x_n, x_m) &\leq D^*(x_n, x_n, x) + D^*(x, x_m, x_m) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \quad (1.16)$$

Hence sequence  $\{x_n\}$  is a Cauchy sequence.  $\square$

*Definition 1.10.* Let  $A$  and  $S$  be two mappings from a  $D^*$ -metric space  $(X, D^*)$  into itself. Then  $\{A, S\}$  is said to be weakly commuting pair if

$$D^*(ASx, SAsx, SAsx) \leq D^*(Ax, Sx, Sx), \quad (1.17)$$

for all  $x \in X$ . Clearly, a commuting pair is weakly commuting, but not conversely as shown in the following example.

*Example 1.11.* Let  $(X, D^*)$  be a  $D^*$ -metric space, where  $X = [0, 1]$  and

$$D^*(x, y, z) = |x - y| + |y - z| + |x - z|. \quad (1.18)$$

Define self-maps  $A$  and  $S$  on  $X$  as follows:

$$Sx = \frac{x}{2}, \quad Ax = \frac{x}{x+2} \quad \forall x \in X. \quad (1.19)$$

Then for all  $x$  in  $X$  one gets

$$\begin{aligned} D^*(SAsx, ASx, ASx) &= \left| \frac{x}{x+4} - \frac{x}{2x+4} \right| + \left| \frac{x}{x+4} - \frac{x}{x+4} \right| + \left| \frac{x}{x+4} - \frac{x}{2x+4} \right| \\ &= \frac{2x^2}{(x+4)(2x+4)} \leq \frac{2x^2}{2x+4} \\ &= \left| \frac{x}{2} - \frac{x}{x+2} \right| + \left| \frac{x}{2} - \frac{x}{x+2} \right| + 0 \\ &= D^*(Sx, Ax, Ax). \end{aligned} \quad (1.20)$$

So  $\{A, S\}$  is a weakly commuting pair.

However, for any nonzero  $x \in X$  we have

$$SAsx = \frac{x}{x+4} > \frac{x}{2x+4} = ASx. \quad (1.21)$$

Thus  $A$  and  $S$  are not commuting mappings.

## 2. The main results

*A class of implicit relation.* Throughout this section  $(X, D^*)$  denotes a  $D^*$ -metric space and  $\Phi$  denotes a family of mappings such that each  $\varphi \in \Phi$ ,  $\varphi : (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+$ , and  $\varphi$  is continuous and increasing in each coordinate variable. Also  $\gamma(t) = \varphi(t, t, a_1 t, a_2 t, t) < t$  for every  $t \in \mathbb{R}^+$  where  $a_1 + a_2 = 3$ .

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*Example 2.1.* Let  $\varphi: (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+$  be defined by

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5). \quad (2.1)$$

The following lemma is the key in proving our result.

**LEMMA 2.2.** *For every  $t > 0$ ,  $\gamma(t) < t$  if and only if  $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$ , where  $\gamma^n$  denotes the composition of  $\gamma$  with itself  $n$  times.*

Our main result, for a complete  $D^*$ -metric space  $X$ , reads as follows.

**THEOREM 2.3.** *Let  $A$  be a self-mapping of complete  $D^*$ -metric space  $(X, D^*)$ , and let  $S, T$  be continuous self-mappings on  $X$  satisfying the following conditions:*

- (i)  $\{A, S\}$  and  $\{A, T\}$  are weakly commuting pairs such that  $A(X) \subset S(X) \cap T(X)$ ;
- (ii) there exists a  $\varphi \in \Phi$  such that for all  $x, y \in X$ ,

$$\begin{aligned} D^*(Ax, Ay, Az) \\ \leq \varphi(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)). \end{aligned} \quad (2.2)$$

Then  $A, S$ , and  $T$  have a unique common fixed point in  $X$ .

*Proof.* Let  $x_0 \in X$  be an arbitrary point in  $X$ . Then  $Ax_0 \in X$ . Since  $A(X)$  is contained in  $S(X)$ , there exists a point  $x_1 \in X$  such that  $Ax_0 = Sx_1$ . Since  $A(X)$  is also contained in  $T(X)$ , we can choose a point  $x_2 \in X$  such that  $Ax_1 = Tx_2$ . Continuing this way, we define by induction a sequence  $\{x_n\}$  in  $X$  such that

$$\begin{aligned} Sx_{2n+1} &= Ax_{2n} = y_{2n}, \quad n = 0, 1, 2, \dots, \\ Tx_{2n+2} &= Ax_{2n+1} = y_{2n+1}, \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.3)$$

For simplicity, we set

$$d_n = D^*(y_n, y_{n+1}, y_{n+1}), \quad n = 0, 1, 2, \dots \quad (2.4)$$

We prove that  $d_{2n} \leq d_{2n-1}$ . Now, if  $d_{2n} > d_{2n-1}$  for some  $n \in \mathbb{N}$ , since  $\varphi$  is an increasing function, then

$$\begin{aligned} d_{2n} &= D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = D^*(Ax_{2n}, Ax_{2n+1}, Ax_{2n+1}) = D^*(Ax_{2n+1}, Ax_{2n}, Ax_{2n}) \\ &\leq \varphi \left( \begin{array}{ccc} D^*(Sx_{2n+1}, Tx_{2n}, Tx_{2n}), & D^*(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}), & D^*(Sx_{2n+1}, Ax_{2n}, Ax_{2n}) \\ D^*(Tx_{2n}, Ax_{2n+1}, Ax_{2n+1}), & & D^*(Tx_{2n}, Ax_{2n}, Ax_{2n}) \end{array} \right) \\ &= \varphi \left( \begin{array}{ccc} D^*(y_{2n}, y_{2n-1}, y_{2n-1}), & D^*(y_{2n}, y_{2n+1}, y_{2n+1}), & D^*(y_{2n}, y_{2n}, y_{2n}) \\ D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}), & & D^*(y_{2n-1}, y_{2n}, y_{2n}) \end{array} \right). \end{aligned} \quad (2.5)$$

Since

$$D^*(y_{2n-1}, y_{2n+1}, y_{2n+1}) \leq D^*(y_{2n-1}, y_{2n-1}, y_{2n}) + D^*(y_{2n}, y_{2n+1}, y_{2n+1}) = d_{2n-1} + d_{2n}, \quad (2.6)$$

hence by the above inequality we have

$$d_{2n} \leq \varphi(d_{2n-1}, d_{2n}, 0, d_{2n-1} + d_{2n}, d_{2n-1}) \leq \varphi(d_{2n}, d_{2n}, d_{2n}, 2d_{2n}, d_{2n}) < d_{2n}, \quad (2.7)$$

a contradiction. Hence  $d_{2n} \leq d_{2n-1}$ . Similarly, one can prove that  $d_{2n+1} \leq d_{2n}$  for  $n = 0, 1, 2, \dots$ . Consequently,  $\{d_n\}$  is a nonincreasing sequence of nonnegative reals. Now,

$$\begin{aligned} d_1 &= D^*(y_1, y_2, y_2) = D^*(Ax_1, Ax_2, Ax_2) \\ &\leq \varphi \left( \begin{array}{cc} D^*(Sx_1, Tx_2, Tx_2), & D^*(Sx_1, Ax_1, Ax_1), D^*(Sx_1, Ax_2, Ax_2) \\ D^*(Tx_2, Ax_1, Ax_1), & D^*(Tx_2, Ax_2, A_2) \end{array} \right) \\ &= \varphi \left( \begin{array}{cc} D^*(y_0, y_1, y_1), & D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2) \\ D^*(y_1, y_1, y_1), & D^*(y_1, y_2, y_2) \end{array} \right) \\ &= \varphi(d_0, d_0, d_0 + d_1, 0, d_0) \\ &\leq \varphi(d_0, d_0, 2d_0, d_0, d_0) = \gamma(d_0). \end{aligned} \quad (2.8)$$

In general, we have  $d_n \leq \gamma^n(d_0)$ . So if  $d_0 > 0$ , then Lemma 2.2 gives  $\lim_{n \rightarrow \infty} d_n = 0$ . For  $d_0 = 0$ , we clearly have  $\lim_{n \rightarrow \infty} d_n = 0$ , since then  $d_n = 0$  for each  $n$ . Now we prove that sequence  $\{Ax_n = y_n\}$  is a Cauchy sequence. Since  $\lim_{n \rightarrow \infty} d_n = 0$ , it is sufficient to show that the sequence  $\{Ax_{2n} = y_{2n}\}$  is a Cauchy sequence. Suppose that  $\{Ax_{2n} = y_{2n}\}$  is not a Cauchy sequence. Then there is an  $\epsilon > 0$  such that for each even integer  $2k$ , for  $k = 0, 1, 2, \dots$ , there exist even integers  $2n(k)$  and  $2m(k)$  with  $2k \leq 2n(k) < 2m(k)$  such that

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \quad (2.9)$$

Let, for each even integer  $2k$ ,  $2m(k)$  be the least integer exceeding  $2n(k)$  satisfying (2.9). Therefore

$$D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) \leq \epsilon, \quad D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) > \epsilon. \quad (2.10)$$

Then, for each even integer  $2k$  we have

$$\begin{aligned} \epsilon &< D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\ &\leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + D^*(Ax_{2m(k)-2}, Ax_{2m(k)-2}, Ax_{2m(k)-1}) \\ &\quad + D^*(Ax_{2m(k)-1}, Ax_{2m(k)-1}, Ax_{2m(k)}) \\ &= D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \end{aligned} \quad (2.11)$$

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So, by (2.10) and  $d_n \rightarrow 0$ , we obtain

$$\lim_{k \rightarrow \infty} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) = \epsilon. \quad (2.12)$$

It follows immediately from the triangular inequality that

$$\begin{aligned} & \left| D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \right| \leq d_{2m(k)-1}, \\ & \left| D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) - D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \right| < d_{2m(k)-1} + d_{2n(k)}. \end{aligned} \quad (2.13)$$

Hence by (2.10), as  $k \rightarrow \infty$ ,

$$\begin{aligned} D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)-1}) & \rightarrow \epsilon, \\ D^*(Ax_{2n(k)+1}, Ax_{2n(k)+1}, Ax_{2m(k)-1}) & \rightarrow \epsilon. \end{aligned} \quad (2.14)$$

Now

$$\begin{aligned} & D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2m(k)}) \\ & \leq D^*(Ax_{2n(k)}, Ax_{2n(k)}, Ax_{2n(k)+1}) + D^*(Ax_{2n(k)+1}, Ax_{2m(k)}, Ax_{2m(k)}) \\ & \leq d_{2n(k)} + \varphi \left( \begin{array}{cc} D^*(Ax_{2n(k)}, Ax_{2m(k)-1}, Ax_{2m(k)-1}), & d_{2n(k)}, D^*(Ax_{2n(k)}, Ax_{2m(k)}, Ax_{2m(k)}) \\ D^*(Ax_{2m(k)-1}, Ax_{2n(k)+1}, Ax_{2n(k)+1}), & d_{2m(k)-1} \end{array} \right). \end{aligned} \quad (2.15)$$

Using (2.14),  $\lim_{k \rightarrow \infty} d_n = 0$ , and continuity and nondecreasing property of  $\varphi$  in each coordinate variable, we have

$$\epsilon \leq \varphi(\epsilon, 0, \epsilon, \epsilon, 0) \leq \varphi(\epsilon, \epsilon, 2\epsilon, \epsilon, \epsilon) = \gamma(\epsilon) < \epsilon \quad (2.16)$$

as  $k \rightarrow \infty$ , which is a contradiction. Thus  $\{Ax_n = y_n\}$  is a Cauchy sequence and hence by completeness of  $X$ , it converges to  $z \in X$ . That is,

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} y_n = z. \quad (2.17)$$

Since the sequences  $\{Sx_{2n+1} = y_{2n+1}\}$  and  $\{Tx_{2n} = y_{2n}\}$  are subsequences of  $\{Ax_n = y_n\}$ ; they have the same limit  $z$ . As  $S$  and  $T$  are continuous, we have  $STx_{2n} \rightarrow Sz$  and  $TSx_{2n+1} \rightarrow Tz$ .

Now consider

$$\begin{aligned} D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n+1}) &= D^*(SAx_{2n-1}, TAx_{2n}, TAx_{2n}) \\ &\leq D^*(SA_{2n-1}, ASx_{2n-1}, ASx_{2n-1}) \\ &\quad + D^*(ASx_{2n-1}, ASx_{2n-1}, ATx_{2n}) \\ &\quad + D^*(ATx_{2n}, ATx_{2n}, TAx_{2n}). \end{aligned} \quad (2.18)$$



Using (ii) and the weak commutativity of  $\{A, S\}$  and  $\{A, T\}$ , we get

$$\begin{aligned}
 & D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n+1}) \\
 & \leq D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) + D^*(ASx_{2n-1}, ATx_{2n}, ATx_{2n}) + D^*(Ax_{2n}, Ax_{2n}, Tx_{2n}) \\
 & \leq D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\
 & \quad + \varphi \left( \begin{array}{cc} D^*(S^2x_{2n-1}, T^2x_{2n}, T^2x_{2n}), & D^*(S^2x_{2n-1}, ASx_{2n-1}, ASx_{2n-1}), \\ & D^*(S^2x_{2n-1}, ATx_{2n}, ATx_{2n}) \\ D^*(T^2x_{2n}, ASx_{2n-1}, ASx_{2n-1}), & D^*(T^2x_{2n}, ATx_{2n}, ATx_{2n}) \end{array} \right) \\
 & \quad + D^*(Ax_{2n}, Ax_{2n}, Tx_{2n}) \\
 & \leq D^*(Sx_{2n-1}, Ax_{2n-1}, Ax_{2n-1}) \\
 & \quad + \varphi \left( \begin{array}{c} D^*(S^2x_{2n-1}, T^2x_{2n}, T^2x_{2n}), D^*(S^2x_{2n-1}, S^2x_{2n-1}, Sx_{2n-1}) \\ \quad \quad \quad + D^*(Sx_{2n-1}, Sx_{2n-1}, Ax_{2n-1}), \\ D^*(S^2x_{2n-1}, TAx_{2n}, TAx_{2n}) + D^*(Tx_{2n}, Tx_{2n}, Ax_{2n}), \\ D^*(T^2x_{2n}, Sx_{2n-1}, Sx_{2n-1}) + D^*(Sx_{2n-1}, Sx_{2n-1}, Ax_{2n-1}), \\ \quad \quad \quad D^*(T^2x_{2n}, TAx_{2n}, TAx_{2n}) + D^*(Tx_{2n}, Ax_{2n}, Ax_{2n}) \end{array} \right) \\
 & \quad + D^*(Ax_{2n}, Ax_{2n}, Tx_{2n}).
 \end{aligned} \tag{2.19}$$

If  $D^*(Sz, Tz, Tz) > 0$ , then as  $n \rightarrow \infty$  we have

$$\begin{aligned}
 & D^*(Sz, Tz, Tz) \\
 & \leq D^*(z, z, z) + \varphi \left( \begin{array}{cc} D^*(Sz, Tz, Tz), & D^*(Sz, Sz, Sz) + 0, D^*(Sz, Tz, Tz) + 0 \\ D^*(Tz, Sz, Sz) + 0, & D^*(Tz, Tz, Tz) + 0 \end{array} \right) + 0 \\
 & \leq \gamma(D^*(Sz, Tz, Tz)) < D^*(Sz, Tz, Tz),
 \end{aligned} \tag{2.20}$$

a contradiction. Therefore,  $Sz = Tz$ .

Now we will prove that  $Az = Sz$ . To end this, consider the inequality

$$D^*(Sx_{2n+1}, Az, Az) \leq D^*(Sx_{2n+1}, ASx_{2n+1}, ASx_{2n+1}) + D^*(Az, Az, ASx_{2n+1}). \tag{2.21}$$

Again using (ii) and the weak commutativity of  $\{A, S\}$ , we have

$$\begin{aligned}
 & D^*(Sx_{2n+1}, Az, Az) \leq D^*(Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}) \\
 & \quad + \varphi \left( \begin{array}{cc} D^*(Sz, Tz, TSx_{2n+1}), & D^*(Sz, Az, Az), D^*(Sz, Az, Az) \\ D^*(Tz, Az, Az), & D^*(Tz, Az, Az) \end{array} \right).
 \end{aligned} \tag{2.22}$$

Taking  $n \rightarrow \infty$ , we have

$$\begin{aligned} D^*(Sz, Az, Az) &\leq D^*(z, z, z) + \varphi \left( \begin{matrix} D^*(Sz, Tz, Tz), D^*(Sz, Az, Az), D^*(Sz, Az, Az) \\ D^*(Tz, Az, Az), D^*(Tz, Az, Az) \end{matrix} \right) \\ &= \varphi(0, D^*(Sz, Az, Az), D^*(Sz, Az, Az), D^*(Sz, Az, Az), D^*(Sz, Az, Az)) \\ &\leq \delta(D^*(Sz, Az, Az)) < D^*(Sz, Az, Az) \end{aligned} \quad (2.23)$$

given there by  $Sz = Az$ . Thus  $Az = Sz = Tz$ . It now follows that

$$D^*(Az, Ax_{2n}, Ax_{2n}) \leq \varphi \left( \begin{matrix} D^*(Sz, Tx_{2n}, Tx_{2n}), & D^*(Sz, Az, Az), D^*(Sz, Ax_{2n}, Ax_{2n}) \\ D^*(Tx_{2n}, Az, Az), & D^*(Tx_{2n}, Ax_{2n}, Ax_{2n}) \end{matrix} \right). \quad (2.24)$$

Then as  $n \rightarrow \infty$ , we get

$$\begin{aligned} D^*(Az, z, z) &\leq \varphi(D^*(Sz, z, z), 0, D^*(Sz, z, z), D^*(z, Az, Az), 0) \\ &\leq \gamma(D^*(Az, z, z)) < D^*(Az, z, z), \end{aligned} \quad (2.25)$$

a contradiction, and therefore  $Az = z = Sz = Tz$ . Thus  $z$  is a common fixed point of  $A, S$ , and  $T$ . The unicity of the common fixed point is not hard to verify. This completes the proof of the theorem.  $\square$

*Example 2.4.* Let  $(X, D^*)$  be a  $D^*$ -metric space, where  $X = [0, 1]$  and

$$D^*(x, y, z) = |x - y| + |y - z| + |x - z|. \quad (2.26)$$

Define self-maps  $A, T$ , and  $S$  on  $X$  as follows:

$$Sx = x, \quad Ax = 1, \quad Tx = \frac{x+1}{2}, \quad (2.27)$$

for all  $x \in X$ .

Let

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5). \quad (2.28)$$

Then

$$A(X) = \{1\} \subset [0, 1] \cap \left[ \frac{1}{2}, 1 \right] = S(X) \cap T(X), \quad (2.29)$$

and for every  $x \in X$ , we have

$$\begin{aligned} D^*(ATx, TAx, TAx) &= D^*(1, 1, 1) = 0 \leq D^*(Ax, Tx, Tx), \\ D^*(ASx, Sx, Sx) &= D^*(1, 1, 1) = 0 \leq D^*(Ax, Sx, Sx). \end{aligned} \quad (2.30)$$

That is, the pairs  $(A, S)$  and  $(A, T)$  are weakly commuting.

Also for all  $x, y, z \in X$ , we have

$$\begin{aligned} D^*(Ax, Ay, Az) &= 0 \\ &\leq \varphi(D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay), D^*(Ty, Ax, Ax), D^*(Ty, Ay, Ay)). \end{aligned} \quad (2.31)$$

That is, all conditions of Theorem 2.3 hold and 1 is the unique common fixed point of  $A, S$ , and  $T$ .

**COROLLARY 2.5.** *Let  $A, R, S, T$ , and  $H$  be self-mappings of complete  $D^*$ -metric space  $(X, D^*)$ , and let  $SR, TH$  be continuous self-mappings on  $X$  satisfying the following conditions:*

- (i)  $\{A, SR\}$  and  $\{A, TH\}$  are weakly commuting pairs such that  $A(X) \subset SR(X) \cap TH(X)$ ;
- (ii) there exists  $a \in \Phi$  such that for all  $x, y \in X$ ,

$$D^*(Ax, Ay, Az) \leq \varphi \left( \begin{array}{c} D^*(SRx, THy, THz), D^*(SRx, Ax, Ax), D^*(SRx, Ay, Ay), \\ D^*(THy, Ax, Ax), D^*(THy, Ay, Ay) \end{array} \right). \quad (2.32)$$

If  $SR = RS, TH = HT, AH = HA$ , and  $AR = RA$ , then  $A, S, R, H$ , and  $T$  have a unique common fixed point in  $X$ .

*Proof.* By Theorem 2.3,  $A, TH$ , and  $SR$  have a unique common fixed point in  $X$ . That is, there exists  $a \in X$ , such that  $A(a) = TH(a) = SR(a) = a$ . We prove that  $R(a) = a$ . By (ii), we get

$$D^*(ARa, Aa, Aa) \leq \varphi \left( \begin{array}{c} D^*(SRRa, THa, THa), D^*(SRRa, ARa, ARa), D^*(SRRa, Aa, Aa), \\ D^*(THa, ARa, ARa), D^*(THa, Aa, Aa) \end{array} \right). \quad (2.33)$$

Hence if  $Ra \neq a$ , then we have

$$\begin{aligned} D^*(Ra, a, a) &\leq \varphi(D^*(Ra, a, a), D^*(Ra, Ra, Ra), D^*(Ra, a, a), D^*(a, Ra, Ra), D^*(a, a, a)) \\ &\leq \varphi(D^*(Ra, a, a), D^*(Ra, a, a), D^*(Ra, a, a), 2D^*(Ra, a, a), D^*(Ra, a, a)) \\ &< D^*(Ra, a, a), \end{aligned} \quad (2.34)$$

a contradiction. Therefore it follows that  $Ra = a$ . Hence  $S(a) = SR(a) = a$ . Similarly, we get that  $T(a) = H(a) = a$ .  $\square$

**COROLLARY 2.6.** *Let  $A_i$  be a sequence self-mapping of complete  $D^*$ -metric space  $(X, D^*)$  for  $i \in \mathbb{N}$ , and let  $S, T$  be continuous self-mappings on  $X$  satisfying the following conditions:*

- (i) there exists  $i_0 \in \mathbb{N}$  such that  $\{A_{i_0}, S\}$  and  $\{A_{i_0}, T\}$  are weakly commuting pairs such that  $A_{i_0}(X) \subset S(X) \cap T(X)$ ;

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(ii) there exists a  $\varphi \in \Phi$  and  $i, j, k \in \mathbb{N}$  such that for all  $x, y \in X$ ,

$$D^*(A_i x, A_j y, A_k z) \leq \varphi \left( \begin{array}{c} D^*(Sx, Ty, Tz), D^*(Sx, A_i x, A_i x), D^*(Sx, A_j y, A_j y), \\ D^*(Ty, A_i x, A_i x), D^*(Ty, A_j y, A_j y) \end{array} \right). \quad (2.35)$$

Then  $A_i, S$ , and  $T$  have a unique common fixed point in  $X$  for every  $i \in \mathbb{N}$ .

*Proof.* By Theorem 2.3,  $S, T$ , and  $A_{i_0}$ , for some  $i = j = k = i_0 \in \mathbb{N}$ , have a unique common fixed point in  $X$ . That is, there exists a unique  $a \in X$  such that

$$S(a) = T(a) = A_{i_0}(a) = a. \quad (2.36)$$

Suppose there exists  $i \in \mathbb{N}$  such that  $i \neq i_0$  and  $j = i_0, k = i_0$ . Then we have

$$D^*(A_i a, A_{i_0} a, A_{i_0} a) \leq \varphi \left( \begin{array}{c} D^*(Sa, Ta, Ta), D^*(Sa, A_i a, A_i a), D^*(Sa, A_{i_0} a, A_{i_0} a), \\ D^*(Ta, A_i a, A_i a), D^*(Ta, A_{i_0} a, A_{i_0} a) \end{array} \right). \quad (2.37)$$

Hence if  $A_i a \neq a$ , then we get

$$\begin{aligned} D^*(A_i a, a, a) &\leq \varphi \left( \begin{array}{c} D^*(a, a, a), D^*(a, A_i a, A_i a), D^*(a, a, a), \\ D^*(a, A_i a, A_i a), D^*(a, a, a) \end{array} \right) \\ &\leq \varphi \left( \begin{array}{c} D^*(A_i a, a, a), D^*(A_i a, a, a), D^*(A_i a, a, a), \\ 2D^*(A_i a, a, a), D^*(A_i a, a, a) \end{array} \right) \\ &< D^*(A_i a, a, a), \end{aligned} \quad (2.38)$$

a contradiction. Hence for every  $i \in \mathbb{N}$  it follows that  $A_i(a) = a$  for every  $i \in \mathbb{N}$ . □

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Shaban Sedghi: Department of Mathematics, Islamic Azad University-Ghaemshahr Branch,  
P.O. Box 163, Ghaemshahr, Iran  
Email address: sedghi\_gh@yahoo.com

Nabi Shobe: Department of Mathematics, Islamic Azad University-Babol Branch, Babol, Iran  
Email address: nabi\_shobe@yahoo.com

Haiyun Zhou: Department of Mathematics, Shijiazhuang Mechanical Engineering University,  
Shijiazhuang 050003, China  
Email address: witman66@yahoo.com.cn